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Refinements of Some Probability Inequalities

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Abstract

In this paper, we present refinements of some probability inequalities. We begin with a brief introduction and mention the well-known Chebyshev's inequality which describes the fundamental relationship between mean and variance of a random variable. Here we first present a more general theorem which besides yielding several interesting results, includes Chebyshev's inequality as a special case.Our next result also includes Chebyshev's inequality as a special case when $\lambda = 0$. Besides, we use these results to prove a generalization as well as a refinement of Chebyshev's inequality. We also present an example which shows that our result gives better bound than Chebyshev's inequality . We also present two theorems, first of which includes Markov's inequality as a special case and the other provides a generalization as well as a refinement of Markov's inequality. Here we also present an example to show that our theorem gives better upper bound than Markov's inequality. We conclude this paper by proving two more results, the first of which provides a generalization as well as a refinement of Chernoff's inequality and the second includes Chernoff's inequality as a special case.

Key words: Probability Inequalities: Chebyshev's inequality, Markov's inequality and Chernoff's inequality. Mean, Variance.

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1 Introduction

Inequalities play a fundamental role in nearly all branches of mathematics,especially so in probability distribution theory. Inequalities are useful for bounding quantities which may be otherwise hard to compute. Probability inequalities are an important instrument which is commonly used practically in all divisions of the theory. Among classical probability inequalities are those by S.N. Bernstein (see,[Petro V.V. [16],III,§3, Theorem 18], Bennet.G [17] and Hoeffding.W [18]).The Bernstein inequality is applicable in case when the probabilities $P(X_j > x)$ decrease exponentially as $x \to \infty$ for all summand forming the sum $\sum_{j=1}^{n} = X_j$.As to the Bennet-Hoeffding inequality, the latter is intended for bounded random variables. The inequality by Yu.V.Prokhorov is adjacent to the Bennet-Heoffding inequality. The probability inequalities are very convenient for estimating the rate of convergence in law of large numbers. There is an edge in probability theory that there lies a probability inequality behind every limit theorem. Since most of probability theory concerns with proving of limit theorems, a number of standard inequalities already exists in the literature(for reference see, DeGroot and Schernish[2], Hogg and Craig[19], S.Ross[8], Gut.A[20], Bertsimas.D and Popescu.I[21], Shoat J.A and Tamarkin J.D[22], J.Smith[24], Y.L.Tong [23], J.V. Uspensky [13]). Several different approaches are to be followed for establishing such type of inequalities.

In recent years several contributions have been made to generalize the probability inequalities under some restrictions on the distribution of the probability(for reference see,(Nagaev S.V[25],[28], [26], [27]), Rosenthal H.P[42], Thachuk S.G[29], Do-ney R.A [30], Spataru A [31], Borovkov A.A[32], Feller.W [33], Pinelis I.F[35], Johnson W.B[34], Ibragimov R and Sharakhmetov Sh.[36], Prokhorov Yu.V([37],[38],[39]), Balaji K. et.al [40], Constantin P. N [41], Steliga. K and Szynal.D[43], Marshall A.W and Olkin.I[44], Rao B.L.S.P [45], Zhou.L and Hu Z.C[46]).The problem of deriving bound on probability that a certain random variable belongs to a given set, given information on some of its moments is very much connected with the development of probability theory in the 20th century. The inequalities due to Chebyshev and Markov are some of the classical and widely used results of the modern theory of probability distribution. Such type of inequalities are useful in structural engineering design decision making.The well-known Chebyshev's inequality describes the fundamental relationship between the mean and the variance of a random variable can be studied as follows:(for reference see, Hogg and Craig,[19] p 55,Petro V.V[16])..

Theorem A. Let the random variable X have a distribution of probability about which we assume only that there is a finite variance σ^2. If μ is the mean of X, then for every $k > 0$,

$$P\Big((|x-\mu| \geq k\sigma)\Big) \leq \frac{1}{k^2}.$$

Theorem A is a special case of following more general results.

Theorem B. Let $u(X)$ be a non-negative function of the random variable X. If $E\big(u(X)\big)$ exists, then for every positive constant c,

$$P\Big(u(X) \geq c\Big) \leq \frac{E\big(u(X)\big)}{c}.$$

Here in this chapter, we first present the following more general theorem which, besides yielding several interesting results includes Theorem A (Chebyshev's inequality) as a special case.

Theorem 1. Let λ be a non-negative real number and let $u(X) \geq \lambda$ be a function of the random variable X. If $E\big(u(X)\big)$ exists, then for every positive constant $c > \lambda$,

$$P\Big(u(X) \geq c\Big) + \frac{\lambda}{c}\Big(1 - P\big(u(X) \geq c\big)\Big) \leq \frac{E\big(u(X)\big)}{c}.$$

Remark 1. For $\lambda = 0$, Theorem 1 reduces to Theorem B.

The following result easily follows from Theorem 1.

Theorem 2. Let λ be a non-negative real number and let $u(X) - \lambda$ be a non-negative function of the random variable X. If $E\big(u(X)\big)$ exists, then for every positive constant $c > \lambda$,

$$P\Big(u(X) \geq c\Big) \leq \frac{E\big(u(X)\big) - \lambda}{c - \lambda}.$$

Proof of Theorem 1. Let X be a random variable of continuous type with p.d.f. f(x), then,

$$E\big(u(x)\big) = \int_{-\infty}^{\infty} u(x)f(x)dx. \tag{1.1}$$

Let

$$A = \big(x; u(x) \geq c\big),$$

then,

$$A^* = \big(x; u(x) < c\big),$$

By hypothesis $u(x) \geq \lambda$, for all x.

Then, we have from (1.1),

$$\begin{aligned} E\big(u(x)\big) &= \int_A u(x)f(x)dx + \int_{A^*} u(x)f(x)dx, \\ &\geq c\left[\int_A f(x)dx\right] + \lambda \int_{A^*} f(x)dx, \\ &= c\left[P\Big(u(X) \geq c\Big)\right] + \lambda\left[1 - \int_A f(x)dx\right], \end{aligned}$$

$$= c\left[P\Big(u(X) \geq c\Big)\right] + \lambda - \lambda\left[P\Big(u(X) \geq c\Big)\right]. \tag{1.2}$$

Or,

$$c\left[P\Big(u(X) \geq c\Big)\right] + \lambda\left[1 - P\Big(u(X) \geq c\Big)\right] \leq E\big(u(x)\big),$$

which gives,

$$\left[P\Big(u(X) \geq c\Big)\right] + \frac{\lambda}{c}\left[1 - P\Big(u(X) \geq c\Big)\right] \leq \frac{E\big(u(x)\big)}{c}.$$

This proves the Theorem 1.

Proof of Theorem 2. From (1.2), we have

$$E\big(u(x)\big) = c\Big[P\big(u(X) \geq c\big)\Big] + \lambda - \lambda\Big[P\big(u(X) \geq c\big)\Big].$$
$$= (c-\lambda)\Big[P\big(u(X) \geq c\big)\Big] + \lambda.$$

This implies

$$(c-\lambda)\Big[P\big(u(X) \geq c\big)\Big] \leq E\big(u(x)\big) - \lambda. \tag{1.3}$$

Since $(c-\lambda) > 0$, from (1.3), we get

$$\Big[P\big(u(X) \geq c\big)\Big] + \lambda \leq \frac{E\big(u(x)\big) - \lambda}{(c-\lambda)},$$

which is the conclusion of Theorem 2.

Remark 2. If we take $u(X) = \big(X-\mu\big)^2 + \lambda$ and $(c-\lambda) = k^2\sigma^2, k \geq 1$,

then

$$P\bigg(\big(X-\mu\big)^2 + \lambda \geq c\bigg) \leq \frac{E\bigg(\big(X-\mu\big)^2 + E(\lambda) - \lambda\bigg)}{k^2\sigma^2},$$

or,

$$P\bigg(\big(X-\mu\big) \geq k^2\sigma^2\bigg) \leq \frac{\sigma^2 + \lambda - \lambda}{k^2\sigma^2},$$

and hence

$$P\bigg(\big|X-\mu\big| \geq k\sigma\bigg) \leq \frac{1}{k^2},$$

which is Chebyshev's inequality.

In Theorem 1,we take $u(X) = \big(X-\mu\big)^2$ and $c = k^2\sigma^2$, we get the following generalization as well as a refinement of Chebyshev's inequality.

Theorem 3. Let the random variable X have a probability distribution function for which the $E(X^2)$ exists. If μ is the mean of X and $\big(X-\mu\big)^2 \geq \lambda > 0$, then for every $k > 0$,

$$P\big(|X-\mu| \geq k\sigma\big) + \frac{\lambda}{k^2\sigma^2}\Big[1 - P\big(|X-\mu| \geq k\sigma\big)\Big] \leq \frac{1}{k^2}.$$

Equivalently

$$P\big(|X-\mu| \geq k\sigma\big) + \frac{\lambda}{k^2\sigma^2}\Big[P\big(|X-\mu| < k\sigma\big)\Big] \leq \frac{1}{k^2}.$$

Similarly if we take $u(X) = \big(X-\mu\big)^2$ and $c = k^2\sigma^2$, in Theorem 3.1.2, we get the following interesting generalization as well as refinement of the Chebyshev's inequality.

Theorem 4. Let the random variable X have a probability distribution function for which the $E(X^2)$ exists. If μ is the mean of X and $\big(X-\mu\big)^2 \geq \lambda > 0$, then for every $k > 0$,

$$P\big(|X-\mu| \geq k\sigma\big) \leq \frac{\sigma^2 - \lambda}{\sigma^2 k^2 - \lambda}.$$

Remark 3. For $\lambda = 0$, Theorem 4 reduced to Chebyshev's inequality.

Remark 4. To show Theorem 4 gives better bound than Chebyshev's inequality, we consider the following example.

Example. Let X be the random variable of discrete type having p.d.f.

$$\begin{aligned} f(x) &= \frac{x}{15}, \quad x = 1,2,3,4,5. \\ &= 0, \quad \text{elsewhere.} \end{aligned}$$

Then,

$$\begin{aligned} \mu = E(X) &= \sum_{x=1}^{5} x f(x) = \sum_{x=1}^{5} \frac{x^2}{15}, \\ &= \frac{1}{15}\big[1+4+9+16+25\big] = \frac{11}{3}. \\ E(X^2) &= \sum_{x=1}^{5} x^2 f(x) = \sum_{x=1}^{5} \frac{x^3}{15}, \\ &= \frac{1}{15}\big[1^3+2^3+3^3+4^3+5^3\big] = 15, \end{aligned}$$

so that variance

$$\sigma^2 = E(X^2) - \big(E(X)\big)^2$$
$$= 15 - \left(\frac{11}{3}\right)^2$$
$$= \frac{135 - 121}{9} = \frac{14}{9}.$$

This gives,

$$\sigma = \frac{\sqrt{14}}{3}.$$

Now,

$X = 1, 2, 3, 4, 5$ and $\mu = \dfrac{11}{3}$, we have

$$|x - \mu| = \left|1 - \frac{11}{3}\right|, \left|2 - \frac{11}{3}\right|, \left|3 - \frac{11}{3}\right|, \left|4 - \frac{11}{3}\right|, \left|5 - \frac{11}{3}\right|$$
$$= \frac{8}{3}, \frac{5}{3}, \frac{2}{3}, \frac{1}{3}, \frac{4}{3}.$$

Therefore,

$$\big|X - \mu\big| \geq \frac{1}{3} \quad \text{for all } x = 1, 2, 3, 4, 5.$$

Now, in Chebyshev's inequality, we take $\mu = \dfrac{11}{3}, \sigma = \dfrac{\sqrt{14}}{3}$ and $k = \dfrac{8}{\sqrt{14}}$, we get

$$P\left(\left|X - \frac{11}{3}\right| \geq \frac{8}{\sqrt{14}}.\frac{\sqrt{14}}{3}\right) \leq \left(\frac{\sqrt{14}}{8}\right)^2,$$
$$P\left(\left|X - \frac{11}{3}\right| \geq \frac{8}{3}\right) \leq \frac{14}{64} = \frac{7}{32}. \tag{1.4}$$

By Theorem 4, we have

$$P\big(|X - \mu| \geq k\sigma\big) \leq \frac{\sigma^2 - \lambda}{\sigma^2 k^2 - \lambda},$$

or

$$P\left(\left|X-\frac{11}{3}\right|\geq\frac{8}{3}\right)\leq\frac{\left(\frac{14}{9}-\frac{1}{3}\right)}{\left(\frac{14}{9}.\frac{64}{14}-\frac{1}{3}\right)}=\frac{11}{61}. \tag{1.5}$$

Since $\frac{11}{61}<\frac{7}{32}$, the bound given by (1.5) is less than the bound

given by (1.4). Hence Theorem 4 gives better upper bound

for $P\big(|X-\mu|\geq k\sigma\big)$ than Chebyshev's inequality.

Our next result follows by taking $u(X)=X$ in Theorem 1.

Theorem 5. For the random variable $X\geq\lambda\geq0$, for which $E(X)$ exists, we have for every $c>\lambda\geq0$,

$$P\big(X\geq c\big)+\frac{\lambda}{c}\Big(1-P\big(X\geq c\big)\Big)\leq\frac{E(X)}{c},$$

or equivalently,

$$P\big(X\geq c\big)+\frac{\lambda}{c}\Big(P\big(X<c\big)\Big)\leq\frac{E(X)}{c}.$$

Taking $\lambda=0$ in Theorem 5, we get the following result which is known as Markov's inequality, (S.Ross[8],pp 408).

Corollary 1. (Markov's inequality). For a non-negative random variable X,

$$P\big(X\geq c\big)\leq\frac{E(X)}{c},\quad\text{for every}\ \ c>0.$$

Remark 5. Theorem 5 provides a generalization as well as refinement of Markov's inequality (Corrolary 1) in case $X\geq\lambda\geq0$.
If we take $u(X)=X$ in Theorem 2, we get the following generalizations as well as a refinement of Markov's inequality.

Theorem 6. For the random variable $X \geq \lambda \geq 0$, for which $E(X)$ exists, we have for every $c > \lambda \geq 0$,

$$P\big(X \geq c\big) \leq \frac{E(X) - \lambda}{c - \lambda}.$$

Remark 6. Markov's inequality follows from Theorem 6 by taking $\lambda = 0$.

Remark 7. To show Theorem 6 is a refinement of Markov's inequality, we consider the following example.

Example. Let X be a random variable of continuous type having p.d.f

$$f(x) = \frac{x}{12}, \quad 1 \leq x \leq 5,$$

$$= 0, \quad \text{elsewhere.}$$

Then,

$$\mu = E(X) = \int_1^5 x.\frac{x}{12}dx$$

$$= \frac{1}{12}\int_1^5 x^2 dx = \frac{1}{12}\left[\frac{x^3}{3}\right]_1^5$$

$$= \frac{1}{36}\big[125 - 1\big] = \frac{31}{9}.$$

Let $c = 4$, by Markov's inequality, we have

$$P\big(X \geq 4\big) \leq \frac{E(X)}{4}$$

$$\leq \frac{\frac{31}{9}}{4} = \frac{31}{36}.$$

We have here $X > 1$, so by our Theorem with $\lambda = 1$,

$$P\big(X \geq 4\big) \leq \frac{E(X) - 1}{4 - 1}$$

$$\leq \frac{\left(\frac{31}{9}\right) - 1}{3} = \frac{22}{27}.$$

Since $\frac{22}{27} < \frac{31}{36}$, Theorem 6 gives better upper bound for $P\big(X \geq 4\big)$ than Markov's inequality.

Let X be a non- negative random variable, then for every $c > 0$

$$P\big(X > c\big) \leq \frac{M(t)}{e^{ct}}, \tag{1.6}$$

where $M(t)$ is the m.g.f of X.
The inequality (1.6) is known as Chernoff's inequality (S.Ross [8], pp 429)). Chernoff's inequality provides the bound that holds for all t simultaneously, so that if we need sharper bound we can use

$$P\big(X \geq c\big) \leq \inf_t \frac{M(t)}{e^{ct}},$$

i.e., we minimize R.H.S of (1.6) over t. It is important to note that the bound in (1.6) is exponentially decreasing with t. Here we use Theorem 2, to present the following generalization as well as a refinement of Chernoff's inequality.

Theorem 7. Let X be a non-negative random variable and let $M(t)$ be the m.g.f of X. If for some non-negative real number $\lambda, \mathrm{Min}_t\big(e^{tX}\big) \geq \lambda$, then for every $c > 0$,

$$P\big(X \geq c\big) + \frac{\lambda}{e^{ct}} P\big(X < c\big) \leq \frac{M(t)}{e^{ct}}, \tag{1.7}$$

or equivalently

$$P\big(X \geq c\big) + \inf_t \left(\frac{\lambda}{e^{ct}} P\big(X < c\big)\right) \leq \inf_t \left(\frac{M(t)}{e^{ct}}\right). \tag{1.8}$$

Proof of Theorem 7. We take $u(X) = e^{tX}$ and replace c by e^{tc} in Theorem 1, it follows that

$$P\big(e^{tX} \geq e^{tc}\big) + \frac{\lambda}{e^{ct}}\Big(1 - P\big(e^{tX} \geq e^{tc}\big)\Big) \leq \frac{E\big(e^{tX}\big)}{e^{ct}}.$$

Or,

$$P\big(X \geq c\big) + \frac{\lambda}{e^{ct}}\Big(1 - P\big(X \geq c\big)\Big) \leq \frac{M(t)}{e^{ct}}.$$

This gives,

$$P\big(X \geq c\big) + \frac{\lambda}{e^{ct}} P\big(X < c\big) \leq \frac{M(t)}{e^{ct}},$$

which proves (1.7). To prove (1.8), we minimize (1.7) over t, we get

$$P\big(X \geq c\big) + \inf_t \left(\frac{\lambda}{e^{ct}} P\big(X < c\big) \right) \leq \inf_t \left(\frac{M(t)}{e^{ct}} \right).$$

Remark 8. For $\lambda = 0$, Theorem 7 reduced to Chernoff's inequality.

Remark 9. Since $\frac{\lambda}{e^{ct}} P\big(X < c\big) \geq 0$, from (1.7), we get

$$P\big(X \geq c\big) \leq \frac{M(t)}{e^{ct}},$$

which is Chernoff's inequality again. This shows that Theorem 3.1.2 is a refinement of Chernoff's inequality.

Next we use Theorem 2, to establish the following more general result which includes Chernoff's inequality as a special case.

Theorem 8. Let X be a non-negative random variable and let $M(t)$ be the m.g.f of X. If for some non-negative real number $\lambda, \min_t \left(e^{tX}\right) \geq \lambda \geq 0$,, then for every $c > 0$,

$$P\big(X \geq c\big) \leq \frac{M(t) - \lambda}{e^{ct} - \lambda}.$$

Proof of Theorem 8. Since $\text{Min}_t\big(e^{tX}\big) \geq \lambda \geq 0$, it follows that $e^{tX} \geq \lambda$ for all t. We take $u(X) = e^{tX}$,
so that,

$$u(X) \geq \lambda \geq 0,$$

then by Theorem 2, we have

$$\begin{aligned} P\big(X \geq c\big) &= P\big(e^{tX} \geq e^{tc}\big) \\ &= \frac{E\big(e^{tX}\big) - \lambda}{e^{ct} - \lambda} \\ &= \frac{M(t) - \lambda}{e^{ct} - \lambda}. \end{aligned}$$

Remark 10. If we take $\lambda = 0$ in Theorem 8, we readily get Chernof's inequality.

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